Branch Completeness in School Mathematics and in Computer Algebra Systems

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Abstract

In many cases when solving school algebra problems (e. g. simplifying an expression, solving an equation), the solution is separable into branches in some manner. The paper describes some approaches to branches that are used in school textbooks and computer algebra systems and compares them with mathematically branch-complete solutions. It tries to identify possible reasons behind different approaches and also indicate some ideas how such differences could be explained to the students.

1 Introduction

In many cases when solving school algebra problems, the solution is separable into branches in some manner. For example, an expression may be undefined in case of some values of the variables (e. g. $1/x$, $\sqrt{x}$) and an equation may have several roots or root groups. In some cases, these branches are explicitly introduced in school mathematics, in other cases the branches may be hidden. This paper examines separable branches of solutions of different problems (simplifications, equations) and completeness of the branch sets. We could say that a solution is mathematically branch-complete if all branches are presented. The paper describes some approaches to branches that are used in school mathematics and computer algebra systems (CASs). It tries to identify possible reasons behind different approaches and also indicate some ideas how such differences could be explained to the students.

The paper is based both on different textbooks (English (e. g. [3], [4]), Russian (e. g. [8]), Norwegian (e. g. [12]), Estonian (e. g. [10], [11])) and on computer algebra systems (Derive 6 [Derive], Maple 8 [Maple], Mathematica 5.2 [Mathematica], Maxima 5.13 [Maxima], MuPAD 4.0 [MuPAD] and TI-92+ [TI-92+], TI-*n*spire [TI-nspire], WIRIS [WIRIS]). Admittedly, these sources do not cover all possible approaches. Different textbooks, other CASs, versions, commands or even some special expressions or equations could work in another way. However, this paper is hopefully adequate enough for most cases. The main aim of the paper is not criticism of a particular CAS or textbook but rather description of a variety of approaches. A special notation is introduced for better overview. Some presented topics are fundamental school topics that are discussed in almost every textbook. Some topics are discussed only in particular textbooks and are not discussed at all or only touched upon briefly in others. At the same time these topics could have recognisable educational potency, particularly when using a CAS. Similarly, some presented examples may occur only in a single CAS, being fairly marginal but still having educational value.

The paper is a part of a larger study (by the author) that examines certain answers produced in CASs that are somewhat unexpected from the point of view of school mathematics. Further separate papers are planned on the topics of infinities and indeterminates, equivalence, and real and complex domains. Therefore, this paper will refrain from deeper discussion on the questions associated with these topics.

There is an overview of the previous related works in Section 2. An approach of evaluation of branching diversities is introduced in Section 3. Section 4 discusses simplification questions and solving of equations is examined in Section 5. Even though other commands may work better in some cases, the usual *Simplify* and *Solve* commands are used first of all. There are some general comments on branching diversities in Section 6 and a conclusion in Section 7.

2 Related works

The author has not found any works that would thoroughly discuss branching in CASs from the viewpoint of school mathematics. This was one reason for the choice of this area for studying. However, there is a great deal of material related to the topic. This section provides only a brief list of such papers.

At first we may mention some comparative overviews that are based on several CASs. Probably the largest experiment aimed at discovering how different CASs solve problems can be found in [17]. However, there are not very many examples from the school in that paper and not many branching examples either. Some interesting examples and comments are provided by [6]. Bernardin says after example $ax = b$: *Often, there seems to be a philosophy among computer algebra systems to return answers even if they do not hold on a finite subset of the parameter space.* The same issues are also discussed in [13]. Stoutemyer lists several theoretical and practical limitations of CASs. Some of them are closely related to branching. His sentence *It is important for users to be aware of some of the limitations of such systems to use them wisely.* is suitable as a slogan for this research.

The paper [9] gives a theoretical overview of branch cuts for complex elementary functions, [2] also discusses complex analysis. It is closely related to one aspect of branching. It is useful to find parallels from further fields of mathematics, for example the idea of using sequents like in Gentzen-type calculi ([7]). It seems that branching in school mathematics is generally not explicitly discussed in papers too often. The importance of branches tends to be emphasized in more specific papers (for instance, on teaching and learning the absolute value [18]).

There are also some papers by this author that are related to the current topic. Some preliminary work for the current article could be found in a paper [14] that tried to classify in relation to correctness, completeness and compactness the answers that have some disturbing qualities when used in the school. A more complete study would require further division of the field. The topic of branches discussed in this paper would be one part of this division. The other notable part would be the topic of real and complex domain [16] that is related to branches as well. For example, in case of the equation $4^x = 64$ Derive (if solution domain is Complex) and MuPAD give branching answers (see Section 5.6). Another topic, which is more loosely related to branches, deals with infinities and indeterminates. However, the branching answers to some fractional equations given by Derive that includes ∞ (see Section 5.6) are discussed in [15].

3 Structure of overview and notation

Before proceeding to the main sections, the structure of the description and specific notation should be explained. There are three matters under consideration – we observe comparatively how branches are treated in school textbooks, in CASs, and what is a mathematically complete branch set. The variations in the first two elements of the above list are the object of this paper. The mathematical branch completeness is a gauge that is explained in every subsection. Unfortunately, in some cases it is not uniquely clear what is the mathematically complete branch set. Special attention in studying the textbooks is paid to the model solutions and also the answers because the expectations for students are mainly presented in these parts. The classic commands (e. g., *Solve* and *Simplify*) are used in CASs at a first approximation.

We determine the *evaluations of branching diversities* (EBD) for each problem type, e. g.,

$$CAS < SCH = MATH.$$

CAS refers to the treatments of branches in computer algebra systems, SCH refers to the treatments of branches in school textbooks, and MATH refers to a mathematically branch-complete solution. The equality sign (=) indicates that branches are similarly presented, the sign $<$ shows that the second treatment is more complete. As different textbooks and CASs may have differing branch sets, it is possible that there is more than one EBD in a particular problem type. The specification is added in parentheses in these cases (e. g., $CAS(1)$) may mean that a multiple root is presented in one time. As the specification is context dependent, there is an explanation for the corresponding EBD. Section 6 includes comments on all found EBDs.

There are more areas where we could find branching, but in this paper we focus on two important areas of school mathematics — simplification of expressions (Section 4) and solving of equations (Section 5). The sections are structured as follows: a general introduction to branching in this area followed by a discussion of a number of more colorful topics. The treatments of branches in textbooks and CASs, and mathematical branch-completeness are explained for each problem type. The evaluations of branching diversities (EBD) are also introduced.

We do not discuss the notation of answers (incl. branches); it is assumed that the notation is understandable for the students.

4 Simplification

4.1 Introduction

Different simplification exercises can be found in textbooks at many places, usually after introduction of a new operation or function. The great majority of the simplification exercises in the textbooks are without (or at least without explicitly presented) branching. Three areas are discussed more thoroughly in this section. The topic of "forbidden branches" is a good example of hidden branches while in the case of expressions with absolute value the branches are (sometimes) explicitly presented. The topic of $\sqrt{a^2}$ is closely related to the absolute value but has independent importance as well.

4.2 "Forbidden branches"

The student learns for a number of operations and functions that operating is impossible (at least at school) in the case of some values of arguments. The first contact with these problems occurs in early grades during subtracting when $2 - 3$ is problematic, because negative numbers are (at least "officially") not yet known to the students. The matter is not discussed in depth in these grades and such expressions are simply avoided. The first commented contact with the "forbidden" operands appears in case of division by zero. The fact that division by zero is undefined is explained by means of multiplication. An important argument is the fact that multiplying by 0 always results in 0.

Explanations are also given for further problematic operations. Thus, the student knows (more or less explained) the following restrictions:

- division by zero is undefined;
- there is no square root for a negative number;
- the domain of a logarithmic function is the set of all positive real numbers;
- the base of logarithm must be positive and must differ from 1;
- tangent function is not defined if $x = (2n + 1)\pi/2$, $n \in \mathbb{Z}$;
- argument of arc sine and arc cosine is from the interval $[-1; 1]$ (not expressly discussed in the school textbooks).

The limits of what is allowed and what is not are fairly clear while calculating with numbers. The situation changes when variables appear in denominators or under square roots or in arguments of other problematic functions. In order to follow these restrictions correctly in mathematical terms one should demonstrate all "forbidden branches" separately in expression transformation exercises. Actually, the distinguishing of forbidden branches is discarded, and the practice is even "legalized". For example, a textbook says:

- *Even though not always explicitly stated, we always assume that variables are restricted so that division by 0 is excluded*;
- *Unless stated to the contrary all variables are restricted so that all quantities involved are real numbers*;
- *The equality is valid only at such variable values where the value of either side of the equality is calculable. For instance, the equality*

$$\frac{x}{x-1} = \frac{x \cdot x}{(x-1)x}$$

is valid only where $x \neq 0$ *and* $x \neq 1$. *As mentioned above, such restrictions are henceforth not explicitly stated in the equalities.*

Such conventions allow the students to remorselessly reduce, expand, isolate variables from the radical, etc., without a thought to division by zero or extracting the square root of a negative number, etc.

The CASs do not show "forbidden branches" either. We look at the expressions where in simplification some parts are cancelled, for example

$$\frac{97x}{x}.$$

All computer algebra systems solve it by giving the answer 97, without recording the peculiarity of $x = 0$. (It is noteworthy that TI-$n$spire adds a warning message: *Domain of the result may be larger*) In addition to the general style of disregard for the special cases, it could be also attributed to automatic simplification that some CASs use. We claim that a result is mathematically branch-complete if "forbidden branches" are also explained. Therefore, EBD is $CAS = SCH < MATH$.

It is another matter whether a computer algebra system (or some other software application) could behave in a more precise manner and separately record special cases. This issue has been examined in [7], who propose to write the results of solution steps in form of sequents (as in Gentzen-type calculi in mathematical logic). It is possible to present only the main branch with the condition(s), for example,

$$x \neq 0, \frac{97x}{x} \Rightarrow 97.$$

Also, it is possible to present all branches separately, for example:

$$x \neq 0, \frac{97x}{x} \Rightarrow 97 \text{ and } x = 0, \frac{97x}{x} \Rightarrow \frac{0}{0}.$$

A case of $\frac{0}{0}$ leads to another area that is discussed in [5]. This question is also discussed in [13] as Sets of Measure Zero. He suggests to introduce the option of simplifying that distinguishes the branches in the conditional form of *if ... then ... else ...* into CASs.

Actually, there are still some specific exercises that emphasize the domain of the expression, for example ([11]), *Find the domain of the expression*

$$\sqrt{\frac{4}{x-1}}.$$

There are also books (e. g. [8]) where (at least in some exercises) the branches are presented. One exercise is presented in Section 4.4. In these cases EBD is $CAS < SCH = MATH$.

A topic of "forbidden branches" is bound up with the topic of infinity-indeterminate and number domains because in cases of different domains there may be different restrictions. For example, $\sqrt{e^z} - e^{z/2}$ should not be simplified when z is complex but should be simplified to 0 when z is real ([2]).

4.3 Absolute value

The topic of absolute value is a classic branched topic (and very educative, as such). The branches are introduced already in definition:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}.$$

There may be exercises that avoid branching by appropriate additional assumption in the text of the exercise: *Simplify expression* $|x+1| - |x-1|$*, where* $-1 \leq x < 1$. Likewise, CASs (except WIRIS) have the possibilities to use such assumptions. For example in Maple:

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simplify(abs(x+1)-abs(x-1)) assuming -1<=x, x<1;
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As all explanations of a branch are equal in all these "parties", EBD is $CAS = SCH = MATH$.

In fact, the school textbooks do not include too many simplifications that contain the absolute value. ([1] is not a regular school textbook but rather a textbook for teacher training.) In case of expression without additional assumptions (e. g. $2 - |x-3|$) the branches spring and textbook [1] presents them.

$$2 - |x-3| = \begin{cases} 2-(x-3) & \text{if } x \geq 3 \\ 2-(-x+3) & \text{if } x < 3 \end{cases} = \begin{cases} 5-x & \text{if } x \geq 3 \\ x-1 & \text{if } x < 3 \end{cases}$$

A separate question would be if this exercise is simplification at all, and the answer to that is more complicated in some sense. CASs do not present branches (at least not in association with the *Simplify* command).

EBD is $CAS < SCH = MATH$.

It is possible to compose miscellaneous expressions that include several absolute values. For example, if absolute values are cancellable CASs give the correct answer, for example, $|x-3| - |3-x|$ is simplified to 0.

4.4 Square root of the square ($\sqrt{a^2}$)

There are quite a few simplification exercises that include expressions in the form of $\sqrt{a^2}$, $(a^2)^{1/2}$, $\sqrt[4]{a^4}$, etc where a is an expression. A question of $\sqrt{a^2}$ is very closely related to the absolute value. On the one hand, the textbooks say that $\sqrt{a^2} = |a|$. On the other hand in simplification exercises some books may somewhat retract and say *If not required separately we do not write* $\sqrt{a^2y} = |a|\sqrt{y}$ *and* $\sqrt[4]{(x-2)^4} = |x-2|$ *but* $\sqrt{a^2y} = a\sqrt{y}$ *and* $\sqrt[4]{(x-2)^4} = x-2$. It is a hidden assumption $a \geq 0$ similar to the one of "forbidden branches".

Derive, Maxima, TI-92+ and TI-*n*spire give $|a|\sqrt{y}$ as an answer in the simplification of $\sqrt{a^2y}$. The other systems do not simplify. This question is also discussed in [2]:

- $\sqrt{z^2}$ *should not simplify, or simplify to* $\operatorname{csgn}(z)z$ *when* z *is complex.*
- $\sqrt{z^2}$ *should not simplify, or simplify to* $\operatorname{sgn}(z)z = |z|$ *when* z *is real.*
- $\sqrt{z^2}$ *should simplify to* z *when* z *is positive.*

When the CAS gives the absolute value (the branches are "compressed") we could (maybe questionably) say $SCH < CAS = MATH$. It is hard to determine EBD when the CAS does not simplify.

There are still books that require presenting branches; for example, in case of

$$2(a+b)^{-1}(ab)^{1/2}\left(1+\frac{1}{4}\left(\sqrt{\frac{a}{b}}-\sqrt{\frac{b}{a}}\right)^2\right)^{1/2}$$

[8] gives branching answer:

$$1 \text{ if } a > 0 \text{ and } b > 0; -1 \text{ if } a < 0 \text{ and } b < 0.$$

It is related to question of ($\sqrt{a^2}$) but also to several issues listed in the section of "forbidden branches". We could say (maybe questionably again) that EBD is $CAS < SCH = MATH$.

4.5 More topics

There are more topics in school mathematics where we could anticipate branching. For example in trigonometry Half-Angle Formulas include $\pm$, e. g.,

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}.$$

However, these expressions are very rare in simplification exercises. Sometimes a half-angle may be in the exercise but it is squared and $\pm$ will be eliminated immediately.

5 Equations

5.1 Introduction

The equations have a central position in school algebra, and branching is essential in case of some equations. It should be noted that here we skip the branchings that are related to complex numbers although they are important, especially in case of the CAS answers (e. g., solutions of the exponent equations). Four areas are discussed more thoroughly — multiplicity of roots, extraneous roots, literal equation and trigonometric equation.

5.2 Multiplicity of roots

Although multiplicity of roots is a wider topic we concentrate here on quadratic equations. In solving the quadratic equation, one can get two different real roots in case of a positive discriminant; such branching is clearly presented both in the textbooks and in CAS answers (EBD is $CAS = SCH = MATH$. A negative discriminant leads to complex numbers and we skip it here. There is a problem of the multiple root if the discriminant is zero (e. g. in case of $x^2 + 2x + 1 = 0$). Some textbooks say that there are two equal roots, some say one real root (a double root) or some say just one real solution. The CASs have applied different approaches as well. Derive, Maxima, TI-92+, TI-$n$spire, MuPAD and WIRIS give a single root while Maple and Mathematica give it twice. It is important to note with respect to the notation that the same signs may have different meanings in different CASs. For example, mark { } means a set in MuPAD but a list in Mathematica. We consider that presenting roots twice is a complete answer, even though it is not particularly important to emphasise this at school because there are no polynomials and equations of the nth degree at school.

As both the textbooks and CASs may present roots once (marked by 1 in EBDs) or twice (marked by 2 in EBDs) the possible EBDs are $CAS(1) = SCH(1) < MATH(2)$, $CAS(2) = SCH(2) = MATH(2)$, $SCH(1) < CAS(2) = MATH(2)$ or $CAS(1) < SCH(2) = MATH(2)$.

5.3 Extraneous roots

It is necessary to consider the branches in the case of the equations (fractional, radical, logarithmic, etc.) that correspond to the expressions with "forbidden branches". Disregarding may result in extraneous roots. Different variants are used in the textbooks to obtain correct final answers. Mainly, checking of potential roots in initial equation is used but detection of domain of the equation can be used as well. Generally, textbooks and CASs give a correct set of solutions (EBD is $SCH = CAS = MATH$) but CASs may be surprisingly "more complete" than would be correct in case of some equations (EBD is $SCH = MATH < CAS$). The CASs (except Maxima and WIRIS) may present by default a real solution that is considered as extraneous at school (e. g., $x = -1$ in case of $\sqrt{2x} = \sqrt{x-1}$ (or $\ln(2x) = \ln(x-1)$). This is related to real and complex domains and will be discussed in a separate paper [16].

The other case can be illustrated by the example where all the systems offer 0 as the answer to the equation

$$\frac{x \cdot x}{x} = 0.$$

TI-$n$spire adds a warning-message *Domain of the result may be larger.* It may be explained by the fact that the original equation is automatically simplified before solving. EBD is $SCH = MATH < CAS$.

5.4 Literal equation

The topic of literal equation is a classic branching topic. The literal equations offer different levels for treatment of branches. The minimal case considered correct in some way would be one where it is assumed by default that the parameter has no "suspicious" values, and only the main branch is calculated. This is often assumed in applied problems (e. g. in physics: $A = P + Prt$; *please express* r). At the next level, the parameter values that result in the branch are recorded with the main branch. The most thorough is the level where all cases are shown separately. We consider the last one as complete.

The behaviour of CASs are criticized in [6]. In case of $ax = b$ he says: *When asked to solve with respect to x, all the systems returned the solution $x = \frac{b}{a}$ even when this answer is obviously not correct for $a = 0$.* He notes that there may be different commands (e. g., *Reduce* in Mathematica) that work better.

Using somewhat later versions of CASs we could say that the CASs apply different approaches. MuPAD records all branches; Derive, Maple, Maxima, TI-92+, TI-$n$spire and WIRIS present only the main branch. In Mathematica, it depends on what command (*Solve*, *Reduce* or *InequalitySolve*) is run. Command *Solve* gives only the main branch, command *Reduce* gives the main branch with the corresponding condition and command *InequalitySolve* gives the complete set of branches. We mark "main-branch-approach" by *1* and "all-branch-approach" by *all*. As both the textbooks and CASs may use both approaches we get plenty of possible EBDs: $CAS(1) < SCH(all) = MATH(all)$, $CAS(all) = SCH(all) = MATH(all)$, $SCH(1) < CAS(all) = MATH(all)$ or $CAS(1) = SCH(all) < MATH(all)$.

5.5 Trigonometric equation

There are different sources for branching in case of the trigonometric equations — periodicity and the families of solutions. The textbooks provide general solutions. For example, in case of the equation

$$\sin x + \cos 2x = 0$$

textbooks give the answer

$$x_1 = \frac{\pi}{2} + 2n\pi \text{ and } x_2 = \frac{\pi}{6} \pm \frac{\pi}{3} + \frac{4}{3}n\pi$$

or

$$x_1 = (-1)^n \, \frac{\pi}{2} + n\pi \text{ and } x_2 = (-1)^{n+1} \, \frac{\pi}{6} + n\pi.$$

However, the CASs work differently. MuPAD gives general solutions; Derive, Maple, Mathematica, TI-92+, TI-*n*spire give (at least by default) only the particular solutions. They have different standards for the choice. This equation is too complicated for Maxima and WIRIS. In case of simpler equations they present particular solutions. It is noteworthy that Mathematica, Maxima, TI-92+, TI-*n*spire adds warning messages when presenting particular solutions, e. g. *Some solutions will be lost* or *Some solutions may not be found*.

As we consider general solution as complete, we get according to CAS approach EBDs: $CAS < SCH = MATH$ or $CAS = SCH = MATH$.

5.6 More topics

In case of equations involving absolute value, the branching takes place according to expressions that are surrounded by the absolute value signs. At least in simpler cases it could be said that EBD is $SCH = CAS = MATH$. There are further interesting examples related to the branches. For instance, Derive may present infinity in solving some fractional equation, for example for

$$\frac{1}{x} = \frac{2}{x} + \frac{1}{x-1}$$

Derive gives

$$x = \pm\infty \vee x = \frac{1}{2}.$$

In case of some exponential equations Derive and MuPAD give branches. For example in case of $4^x = 64$ Derive gives

$$x = 3 + \frac{3 \cdot \pi \cdot i}{\ln 2} \vee x = 3 - \frac{2 \cdot \pi \cdot i}{\ln 2} \vee x = 3 + \frac{2 \cdot \pi \cdot i}{\ln 2} \vee x = 3 - \frac{\pi \cdot i}{\ln 2} \vee x = 3 + \frac{\pi \cdot i}{\ln 2} \vee x = 3$$

and MuPAD gives

$$\left\{3 + \frac{\pi \cdot k \cdot 2 \cdot i}{\ln 4} \mid k \in \mathbb{Z}\right\}.$$

The paper [6] marks problems with equations like

$$\frac{1}{\tan x} = 0.$$

These problems are more completely discussed in the paper [16].

6 Comments on branching diversities

Theoretically we could compose quite many different EBDs while practically only some of them are present. The situations where a textbook and CAS give the same amount of branches ($CAS = SCH = MATH$) are not very interesting from the perspective of this paper as the student (and also the teacher) gets the answer that accords with school presentation and mathematics. The Tables 1 and 2 contain all other abovementioned EBDs with references. The

EBD	Problem type
$CAS = SCH < MATH$	Forbidden branches are not recorded (CAS, SCH)
$CAS < SCH = MATH$	Forbidden branches are recorded (SCH), not recorded (CAS) Absolute value, all branches (SCH)
$SCH < CAS = MATH$	$\sqrt{a^2} \to a$ (SCH)

Table 1: EBDs (Simplification)

EBD	Problem type
$CAS = SCH < MATH$	Multiplicity of roots 1 (CAS, SCH) Literal equation 1 branch (CAS, SCH)
$CAS < SCH = MATH$	Multiplicity of roots 1 (CAS), 2 (SCH) Literal equation 1 branch (CAS), all branches (SCH) Particular solution of trigonometric equation (CAS)
$SCH < CAS = MATH$	Multiplicity of roots 1 (SCH), 2 (CAS) Literal equation 1 branch (SCH), all branches (CAS)
$SCH = MATH < CAS$	Extraneous roots

Table 2: EBDs (Equations)

different types of evaluations of branching diversities that we found are briefly commented in this section.

The cases where $SCH = MATH < CAS$ are deficiencies of CASs. The situation $CAS = SCH < MATH$ needs some comments. The explanation-justification related to hidden forbidden branches is presented (maybe too modestly) in textbooks. It may be assumed that branch completeness is rejected for the sake of compactness. Apparently, it is more complicated as well as more time- and space-consuming to (repeatedly) record several branches and special cases. Furthermore, repeated recording entails the danger of oversights, etc. In addition, it is more difficult to grasp the answer where it contains many special cases and branches, which distract attention from the main line.

In other cases, a CAS may present a complete set of branches while a textbook presents an incomplete set or vice versa. If CAS presents a complete and textbook an incomplete set ($SCH < CAS = MATH$), there may be the explanation-justification in textbook ($\sqrt{a^2} \to a$) or it is possible to use the same explanation as in the textbooks that present the complete set of branches (multiple root, literal equation).

If a textbook is more complete ($CAS < SCH = MATH$), there are three possible manners: avoid using CAS (simplification of an expression involving absolute value), try to find a complete set manually (or with CAS)(trigonometric equation), or explain that a (more) complete set of branches is not necessary for school (multiple root, literal equation).

7 Conclusion, further work

Even though all possibilities are not observed, one may notice that the branch sets vary in cases of different textbooks and CASs. We described (sometimes maybe questionably) the mathematically correct and complete branch sets and compared the approaches of textbooks and CASs and the mathematically complete sets. We classified the situations and found four types that were briefly commented. The use of CASs in teaching and learning the branch-related topics seems to be reasonable. Almost all diversities are explainable. However, more studying is needed to elaborate a detailed framework. Hopefully, the teachers (and others) could then place their own examples into this framework and get useful information to improve their work.

There are several open research "branches" that could ensue from this paper. For example, a study of teachers attitudes to branching would be very interesting. What approaches are more suitable for teaching and learning? The usual commands *Simplify* and *Solve* with default settings are used in above examples. The other possibilities (e. g. *assuming* in Maple, or *Reduce* in Mathematica) are only touched. Actually, the proper use of different commands, assumptions and settings could make CAS more suitable for school mathematics. For example, there are special tools for determination of real or complex domain. A wide variety of means in different CASs is the object of further study.

There are also interesting questions about automatic simplification. Also, the warning messages are a possible object of future work — are the messages adequate? Are they helpful for students? It is possible to compose papers for specific mathematical topics, e. g., absolute value, literal equations, trigonometry.

Hopefully, it will help students and teachers use CASs more wisely.

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Software Packages

[Derive] http://www.derive-europe.com/

[Maple] http://www.maplesoft.com/

[Mathematica] http://www.wolfram.com/

[Maxima] http://maxima.sourceforge.net/

[MuPAD] http://www.mupad.de/

[TI-92+] http://education.ti.com/educationportal/sites/US/productDetail/us_ti92p.html

[TI-nspire] http://www.ti-nspire.com/tools/nspire/index.html

[WIRIS] http://www.wiris.com/